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THE REFRACTION OF A SHEAR WAVE IN A NON-LINEARLY ELASTIC AND ELASTOPLASTIC HALF-SPACE*

A.G. BYKOVTSEV

Dynamic equations describing antiplane deformation in a non-linearly elastic medium are studied. Selfsimilar solutions are analysed for the case when the displacement rates and stresses depend on two variables $x = x_1 - ct, y = x_2$. In the limiting case, when the $\tau - \gamma$ -plot for a non-linearly elastic material is the same as that for a perfect elastoplastic material, a solution is constructed for the problem of refraction of plane-polarized plane waves of pure shear in a non-linearly elastic half-space. The results obtained are compared with the solution constructed earlier /1/ in which the system of Prandtl-Reuss equations was used to study the refraction of pure shear waves in an elastoplastic half-space.

Selfsimilar problems in which the displacement rates and stresses depend only on the ratio of the coordinates, were studied in /2–7/.

1. Let us consider the dynamic problem of the theory of complex shear in a non-linearly elastic medium. In a Cartesian coordinate system x_i the displacement vectors u and displacement rate vectors w are directed along the x_3 axis, and depend only on x_1, x_2 and the time t. All components of the stress tensor except $\tau_1 = \sigma_{13}(x_1, x_2, t)$ and $\tau_2 = \sigma_{23}(x_1, x_2, t)$ vanish. In this case the equations of motion have the form

$$\frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} - \rho \frac{\partial^2 u}{\partial t^2} = 0 \tag{1.1}$$

Let us consider the selfsimilar solutions of (1.1) depending only on $x = x_1 - ct,\ y = x_2$. The relations of the non-linear theory of elasticity for antiplane deformations take the form

$$\tau_1 = f(\gamma) u_{,x}, \quad \tau_2 = f(\gamma) u_{,y}, \quad \gamma = \sqrt{(u_{,x})^2 + (u_{,y})^2} \tag{1.2}$$

The relations of the deformation theory of plasticity /8/ are the same as (1.2) under an active load $\gamma_{,t} \geqslant 0$ and differ from them during unloading when $\gamma_{,t} < 0$. According to deformation theory τ and γ are connected by a linear relation during the unloading.

From (1.1) and (1.2) it follows that the displacement u satisfies the non-linear wave equation

$$\gamma f(\gamma)(u_{,xx} + u_{,yy}) + f'(\gamma)(u_{,xx}(u_{,x})^2 + 2u_{,xy}u_{,x}u_{,y} + u_{,yy}(u_{,y})^2) - \gamma\rho c^2 u_{,xx} = 0 \tag{1.3}$$

The last relation of (1.2) will be satisfied identically if we put

$$u_{,x} = \gamma \sin\theta, \quad u_{,y} = \gamma\cos\theta \tag{1.4}$$

Substituting these expressions into (1.3) and writing the resulting equation together with the conditions of compatibility of deformations, we obtain the following system of equations for determining γ and θ:

$$\begin{gathered}\gamma_x (f(\gamma) + \gamma f'(\gamma) - \rho c^2)\sin\theta + \gamma_{,y}(f(\gamma) + \gamma f'(\gamma))\cos\theta + \\ \theta_{,x}\gamma(f(\gamma) - \rho c^2)\cos\theta - \theta_{,y}\gamma f(\gamma)\sin\theta = 0 \\ \gamma_{,x}\cos\theta - \gamma_{,y}\sin\theta - \theta_{,x}\gamma\sin\theta - \theta_{,y}\gamma\cos\theta = 0\end{gathered} \tag{1.5}$$

System (1.5) is hyperbolic when

$$f(\gamma)(\rho c^2 - \gamma f'(\gamma) + f(\gamma)) + \gamma f'(\gamma)\rho c^2\cos^2\theta = r^2 > 0$$

The characteristics and the relations along them in this case have the form

$$dy(\gamma f'(\gamma)\sin^2\theta + f(\gamma) - \rho c^2) = dx(\gamma f'(\gamma)\sin\theta\cos\theta \pm r) \tag{1.6}$$

$$d\theta\gamma(\rho c^2\cos^2\theta - f(\gamma)) + d\gamma(\rho c^2\cos\theta\sin\theta \pm r) = 0 \tag{1.7}$$

2. Let us consider a limiting case, when the $\tau - \gamma$ plot for the non-linearly elastic material is the same as that for a perfectly elastoplastic material, i.e. when the function

*Prikl.Matem.Mekhan.,50,3,490-497,1986

$f(\gamma)$ has a kink point. In this case we shall use the relations obtained to study the

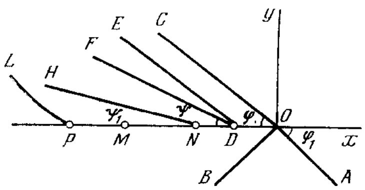

Fig.1

refraction of plane-polarized shear waves passing from an elastic half-space with parameters $\mu_1, \rho_1, a_1 = \sqrt{\mu_1/\rho_1}$ into an elastoplastic half-space (EPH) with parameters $\mu_2, \rho_2, a_2 = \sqrt{\mu_2/\rho_2}, k$. In the case of the limit model of a non-linearly elastic body in the elastic domain $f(\gamma) = \mu = \mathrm{const}$, and in the plastic domain the yield condition $\tau_1^2 + \tau_2^2 = k^2$ must hold. From this it follows that $f(\gamma) = k\gamma^{-1}$.

Let a plane wave OA (Fig.1) impinge on the boundary surface $y = 0$. The equation of the incident wavefront at any instant of time has the form $y\cos\varphi_1 + x\sin\varphi_1 = -\omega_1 = \mathrm{const}$, and we have the following relations behind the incident wavefront (φ_1 is the angle of incidence)

$$\tau_i = \tau_i(\omega_1), \quad i = 1,2, \; w = w_1(\omega_1), \quad w = u_{,t} = -c\gamma\sin\theta \tag{2.1}$$

The equation of the reflected wavefront has the form $y\cos\varphi_1 - x\sin\varphi_1 = \omega_2 = \mathrm{const}$. The solution behind the reflected wavefront in the elastic half-space is obtained by combining the solution (2.1) with the solution for the reflected wave, which has the form

$$\tau_i = \tau_i(\omega_2), \quad i = 1, 2, \quad w = w_2(\omega_2) \tag{2.2}$$

The equation of the refracted wavefront at any instant of time has the form $y\cos\varphi + x\sin\varphi = \omega = \mathrm{const}$. We have the following relations behind the refracted wavefront in E:

$$\tau_i = \tau_i(\omega), \quad i = 1, 2, \quad w = w(\omega) \tag{2.3}$$

where (φ the angle of refraction such that $a_1\sin\varphi = a_2\sin\varphi_1$).

From (1.1), (2.3) it follows that in the elastic domain behind the refracted wave

$$\tau_1 = \mu_2\gamma\sin\theta, \quad \tau_2 = \mu_2\,\mathrm{ctg}\,\varphi\gamma\sin\theta \tag{2.4}$$

From (1.1), (2.1), (2.2) we have

$$\tau_1(\omega_i) = -\mu_1 c^{-1} w_i(\omega_i), \quad \tau_2(\omega_i) = (-1)^i \mu_1 c^{-1}\,\mathrm{ctg}\,\varphi_1 w_i(\omega_i) \tag{2.5}$$
$$i = 1, 2$$

At the boundary $y = 0$ the normal stress τ_2 and displacement rate w are assumed to be continuous; this implies that

$$w(x) = w_1 + w_2, \quad \tau_2(x) = \mu_1 c^{-1}\,\mathrm{ctg}\,\varphi_1(w_2 - w_1) \tag{2.6}$$
$$w_i = w_i(-x\sin\varphi_1), \; i = 1, 2$$

Here $w(x)$ is the rate of displacement and $\tau_2(x)$ is the stress at the boundary in the EPH. Eliminating from Eq.(2.6) the function $w_2(-x\sin\varphi_1)$, we obtain the boundary condition for the EPH in the form

$$2w_1(-x\sin\varphi_1) = w(x) - c\mu_1^{-1}\,\mathrm{tg}\,\varphi_1\tau_2(x) \tag{2.7}$$

Henceforth we shall assume that the function $w_1(\omega_1)$ is given, i.e. the profile and intensity of the incident wave are known.

Let us consider the refracted wave in the EPH. The material is assumed to be at rest $w = u = \tau_1 = \tau_2 = 0$, in front of the refracted wavefront OC, i.e. the material will be in the elastic state near OC. We have $\gamma = 0$ on the line OC, and from (1.6), (1.7) we obtain

$$x - \varkappa y = \mathrm{const}, \quad \gamma(\cos\theta - \varkappa\sin\theta) = 0 \tag{2.8}$$
$$x + \varkappa y = \mathrm{const}, \quad \gamma(\cos\theta + \varkappa\sin\theta) = \mathrm{const} \tag{2.9}$$
($\varkappa = \sqrt{M^2 - 1}$, $M = ca_2^{-1}$ is the Mach No.

The constant on the right-hand side of relation (2.8) is the same for all characteristics, and should therefore be regarded as the integral of the equation of motion in the elastic region. Using relations (2.8), we can write the condition at the boundary (2.7) in the form

$$2w_1(-x\sin\varphi_1) = -c\gamma\sin\theta\left(1 + \frac{\mu_2\,\mathrm{tg}\,\varphi_1}{\mu_1\,\mathrm{tg}\,\varphi}\right) \tag{2.10}$$

From the integrals (2.8), (2.9) it follows that in the elastic domain, θ has a constant value, and γ remains constant along the characteristics (2.9). This implies that the yield point will be attended at once along the whole characteristic, provided that it is attained at least one point.

The condition for attaining the yield point has the form

$$\tau_1^2 + \tau_2^2 = k^2 = \mu_2^2\gamma^2, \quad \gamma = \pm k\mu_2^{-1} \tag{2.11}$$

Substituting the second relation of (2.11) into (2.4), we obtain

$$\sin\theta = \pm \sin\varphi \tag{2.12}$$

Before continuing with the construction of the solution, we must decide on the choice of sign in expressions (2.11), (2.12). This can be done as follows. In the rectangular coordinate system xOy the displacement u and displacement rate w in the EPH (Fig.1) for the problem in question, and we have $w = u = 0$ at the point O. Consequently, when $x < 0$, we have $\partial u/\partial x \leqslant 0$ and relations (1.4) imply that γ and $\sin\theta$ are opposite in sign. We find from (2.4) that $\tau_1 \leqslant 0, \tau_2 \leqslant 0$. At the point D, at which the yield point is first reached, we have $\tau_1 = k\sin\theta$, $\tau_2 = k\cos\theta$. Therefore in the elastic domain $\theta = \pi + \varphi$ and the plus sign should be chosen in relation (2.11).

From (2.10)–(2.12) it follows that the material will remain elastic between the characteristics OC and DE (Fig.1) until the following equality is reached at the same point D at the boundary:

$$2\left|w_1^*(-x\sin\varphi_1)\right| = \frac{k}{\sqrt{\mu_2\rho_2}}\left(1 + \frac{\mu_2\operatorname{tg}\varphi_1}{\mu_1\operatorname{tg}\varphi}\right) \tag{2.13}$$

The material is in the plastic state to the left of the line DE and Eqs.(1.6), (1.7) yield

$$dy\left(\cos\theta - \Delta\sqrt{\frac{w}{-\sin\theta}}\right) = -\sin\theta\, dx \tag{2.14}$$

$$I(0,\theta) - 2\Delta\sqrt{w} = \text{const}$$

$$dy\left(\cos\theta + \Delta\sqrt{\frac{w}{-\sin\theta}}\right) = -\sin\theta\, dx \tag{2.15}$$

$$I(\theta, \pi+\varphi) + 2\sqrt{M} = 2\Delta\sqrt{w}, \quad \Delta = \sqrt{\frac{\rho_2 c}{k}}$$

$$I(a,b) = \int_a^b \frac{d\theta}{\sqrt{-\sin\theta}}$$

Since on the line DE $w = k(\mu_2\rho_2)^{-1/2}$, $\theta = \pi + \varphi$ and the characteristics of (2.15) intersect the line DE, the constant appearing in the relation along the characteristic of this family is the same for all characteristics. Therefore, the second equation of (2.15) should be regarded as an integral of the equations of motion in the plastic domain. From the integrals (2.14), (2.15) we find, that along every characteristic of the family (2.14) θ and w remain constant, and from this it follows that the characteristics of (2.14) are rectilinear. Thus we have

$$y\left(\Delta\sqrt{\frac{w}{-\sin\theta}} - \cos\theta\right) - x\sin\theta = \text{const} \tag{2.16}$$

$$I(0,\theta) - 2\Delta\sqrt{w} = \text{const}$$

The characteristics (2.16) intersect the lines DE and are inclined to the x axis at an angle Ψ, for which

$$\operatorname{tg}\Psi = -\frac{\sin\varphi}{M + \cos\varphi} < \operatorname{tg}\varphi$$

Thus we have for the equations of motion in the plastic domain on the line DE the Cauchy problem. Solving this problem we determine w and θ between the characteristic DE in the elastic domain, and the characteristic DF in the plastic domain where $w = k(\mu_2, \rho_2)^{-1/2}$, $\theta = \pi + \varphi$.

Eliminating the function w from the boundary condition (2.7) with the help of the integral (2.15), we obtain

$$2w_1(-x\sin\varphi_1) = W(\theta) - c\mu_1^{-1}\operatorname{tg}\varphi_1 k\cos\theta \tag{2.17}$$

$$W(\theta) = \left(\int_\theta^{\pi+\varphi} \frac{d\theta}{2\Delta\sqrt{-\sin\theta}} + \frac{\sqrt{M}}{2\Delta}\right)^2$$

Let $\theta = \theta_1$ be the root of this equation. Then from (2.15) we obtain the value of $w = W(\theta_1)$ at the boundary $y = 0$.

The values of θ and w remain constant along the line

$$y\left(\Delta\sqrt{\frac{w(\theta_1)}{-\sin\theta_1}}-\cos\theta_1\right)-(x-x_N(\theta_1))\sin\theta_1=0 \tag{2.18}$$

(x_N is the coordinate of the point on the boundary at which $\theta=\theta_1$). The angle of inclination of this characteristic to the x axis is connected with θ_1 by the relation

$$\operatorname{tg}\Psi_1=-\sin\theta_1\left(\Delta\sqrt{\frac{w(\theta_1)}{-\sin\theta_1}}-\cos\theta_1\right)^{-1} \tag{2.19}$$

In order for the solution constructed to exist, it is necessary that the angle Ψ_1 should increase when the point N moves along the x axis, and that condition of loading should hold in the plastic domain. The condition of loading in the present case has the form

$$\gamma_{,t}=-c\gamma_{,x}=-\frac{1}{w(\theta_1)}\frac{\partial w(\theta_1)}{\partial x}+\operatorname{ctg}\theta_1\frac{\partial\theta_1}{\partial x}\geqslant 0 \tag{2.20}$$

Using the integral (2.15), we can write the condition of loading in the form

$$\frac{\partial\theta_1}{\partial x}(\operatorname{ctg}\theta_1+[\Delta\sqrt{-\sin\theta}\,(I(\theta_1,\pi+\varphi)+\Delta\sqrt{M})]^{-1})\geqslant 0 \tag{2.21}$$

i.e. the condition of loading will hold as long as $\partial\theta_1/\partial x\geqslant 0$.

Differentiating relation (2.19) with respect to x, we find that the angle Ψ_1 will increase as we move along the x axis, provided that

$$\left[\frac{3}{2}-\frac{\cos\theta_1}{\sqrt{-\sin\theta_1}}\left(\frac{1}{2}I(\theta,\pi+\varphi)+\sqrt{M}\right)\right]\frac{\partial\theta_1}{\partial x}\geqslant 0 \tag{2.22}$$

i.e. the condition that Ψ_1 increases will hold as long as $\partial\theta_1/\partial x\geqslant 0$.

Since θ_1 satisfies Eq.(2.17), differentiating the latter with respect to w_1 yields

$$2\Delta^2\sqrt{-\sin\theta_1}=$$
$$-\frac{\partial\theta_1}{\partial w_1}\left(\frac{1}{2}I(\theta_1,\pi+\varphi)+\sqrt{M}+\sqrt{-\sin\theta_1}\operatorname{tg}\varphi_1\frac{c^2\rho_2}{\mu_1}\right)$$

From this we find that the quantity θ_1 decreases as w_1 increases, and this implies that $\partial\theta_1/\partial x\geqslant 0$ and conditions (2.21), (2.22) will hold as long as $w_1(\omega_1)$ increases as ω_1 increases. We note that in this case the monotonicity of the functions appearing in (2.17) implies that the latter will always have a unique solution when $\theta\in[\pi,\pi+\varphi]$. When θ_1 decreases, it can reach the value π when the characteristic (2.14) in the plastic domain becomes parallel to the x axis. This will happen when the amplitude of the incident wave w_1 attains the value

$$w_1{}^0=\frac{k}{2\sqrt{\mu_1\rho_1}}\left[\frac{1}{\cos\varphi_1}+\frac{\rho_1a_1}{\rho_2a_2}\left(1+\sqrt{\sin\varphi}\int_0^{\varphi}\frac{d\varphi}{\sqrt{\sin\varphi}}\right)^2\right] \tag{2.23}$$

Let this value be reached at the point M, and let $w_1(\omega_1)$ increase from that moment. Then the line MP will become a characteristic and we shall have on it

$$\theta_1=\pi,\ \tau_1=0,\quad \tau_2=-k,\quad w=w_*$$
$$w_*=\frac{1}{4\Delta^2}\left(\int_0^{\varphi}\frac{d\varphi}{\sqrt{\sin\varphi}}+2\sqrt{M}\right)^2$$

The solution in EPH is determined by the boundary condition on the line OM. The line MP is a stationary line of discontinuity on which the displacement rates undergo a jump, and the dynamic conditions of compatibility on the surface of strong discontinuity imply that the stress τ_2 is continuous on the line MP.

In the case when $w_1(\omega_1)>w_1{}^0$, the condition of loading (2.21) holds also in the plastic domain, but special attention should be given to this condition when $y=0$, i.e. in the slippage zone. The condition of loading holds on the stationary line of velocity discontinuity when $w_e>w_p$ where w_e is the displacement rate in the elastic domain, and w_p in the plastic domain. From (2.6) and (2.15) it follows that the condition of loading holds when

$$2w_1(-x\sin\varphi_1)-k(\sqrt{\mu_1\rho_1}\cos\varphi_1)^{-1}>w_* \tag{2.24}$$

Thus, if the profile of the incident wave does not exceed $w_1{}^0$, the condition of loading will hold as long as the function $w_1(\omega_1)$ increases, and unloading begins when the maximum value of the profile has been passed. If the profile of the incident wave exceeds $w_1{}^0$, then slippage (discontinuity of displacement) occurs in the corresponding zone at the boundary separating two media. In this case the condition of loading will hold as long as the profile exceeds $w_1{}^0$, and unloading will begin when the profile of the incident wave becomes smaller than w_1^0.

We note that the slippage zone represents, according to the terminology of the mechanics of fracture, a slippage crack moving along the boundary separating two media. The displacement u becomes strongly discontinuous in the slippage zone. The fact which appears to be essential is, that the zone of slippage, its formation and development, are all described by the equations of dynamics of elastoplastic media, without bringing in the physical laws of the mechanics of fracture. The stresses and deformations at the crack tip are finite. Thus the model of a perfectly elastoplastic body enables us, at least in the case in question, to carry out a closed investigation of the development of the slippage cracks as surfaces of stationary discontinuities.

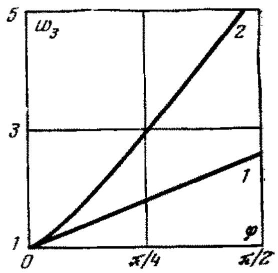

Fig.2

It is interesting to compare the solution constructed with the results obtained in /1/, where the problem of the refraction of plane-polarized, plane shear waves at the boundary separating the elastic and elastoplastic half-spaces was solved using the system of Prandtl-Reuss equations. When $\gamma \leqslant k\mu_2^{-1}$, the solutions based on the model of a perfectly elastoplastic body (the theory of plastic flows) are the same as those based on the limit model of a non-linearly elastic body (the deformation theory of plasticity), therefore the elastoplastic boundaries are also the same in both cases. The difference appears in the plastic zone, and this leads to different conditions of slippage. For a perfectly elastoplastic model the condition has the form /1/

$$w_1^\circ = \frac{k}{2\sqrt{\mu_1\rho_1}}\left(\frac{\rho_1 a_1}{\rho_2 a_2}(1+\varphi) + \frac{1}{\cos\varphi_1}\right)$$

The dependence of the quantity $w_3 = (2\sqrt{\mu_1\rho_1}k^{-1}w_1^\circ - (\cos\varphi_1)^{-1})\,\rho_2 a_2/(\rho_1 a_1)$ on the angle of refraction φ is shown in Fig.2, according to the perfect elastoplastic body model (curve *1*) and the limit model of the non-linearly elastic body (curve *2*).

It should also be noted that in case of slippage the coordinates of the point of the body x_p, from which the unloading wave begins to propagate, obtained using the above models, are also different.

3. Let us consider the propagation of the unloading wave (UW). We note that if the unloading follows the model of a non-linearly elastic body, the angle θ increases as $w_1(\omega_1)$ increases, and the angle of inclination of the characteristics (2.14) to the x axis will increase in the unloading zone and a shock UW will form.

Henceforth, we shall proceed according to the deformation theory of plasticity, i.e. we shall assume that the unloading takes place linearly and *PL* is the line separating the plastic domain from the unloading zone. The method of determining the initial velocity of UW used in /1/ can be generalized and used to find the velocity of the UW at any of its points.

The relations (2.14), (2.15) which hold in the plastic domain, can be written in the form

$$4\Delta\sqrt{w} = \int_\theta^{\pi+\varphi}\frac{d\theta}{\sqrt{-\sin\theta}} + 2\sqrt{M} - \int_0^\theta\frac{d\theta}{\sqrt{-\sin\theta}} - f_3\left(\frac{y}{c_p} - x\right) \tag{3.1}$$

$$2\int_0^\theta\frac{d\theta}{\sqrt{-\sin\theta}} = f_3\left(\frac{y}{c_p} - x\right) + \int_0^{\pi+\varphi}\frac{d\theta}{\sqrt{-\sin\theta}} - 2\sqrt{M} \tag{3.2}$$

($c_p = \sqrt{-\sin\theta}\sin\theta\,(\Delta\sqrt{w} - \sqrt{-\sin\theta}\cos\theta)^{-1}$ is the velocity of plastic waves.

The boundary condition (2.17) has the following form at the boundary:

$$2w_1(-x\sin\varphi_1) = \frac{1}{16\Delta^2}\left[\int_0^{\pi+\varphi}\frac{d\theta}{\sqrt{-\sin\theta(x,0)}} + 2\sqrt{M} - f_3(-x)\right]^2 - \frac{c}{\mu_1}\operatorname{tg}\varphi_1 k\cos\theta(x,0) \tag{3.3}$$

Differentiating Eqs.(3.2) and (3.3) with respect to x with $y=0$, we obtain a system of equations for determining $\partial\theta(x,0)/\partial x$, $f_3'(-x)$, and the system yields

$$\begin{gathered} f_3'(-x) = 2R_1(-x\sin\varphi_1)\,F(\theta(x,0)) \\ F(\theta(x,0)) = \left[\frac{1}{2\Delta^2}F_1(\theta(x,0)) - \frac{c}{\mu_1}\operatorname{tg}\varphi_1 k\sin\theta(x,0)\times\sqrt{-\sin\theta(x,0)}\right]^{-1}, \quad F_1(\theta(x,0)) = \int_\theta^{\pi+\varphi}\frac{d\theta}{\sqrt{-\sin\theta(x,0)}} + 2\sqrt{M} \\ R_1(-x\sin\varphi_1) = -\sin\varphi_1 w_1'(-x\sin\varphi_1) \quad \text{when} \quad x \geqslant x_p \end{gathered} \tag{3.4}$$

Differentiating Eq.(3.2) with respect to x with $y = y(x)$, i.e. on the UW, we obtain

$$\frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial y} c^* = \left(1 - \frac{c^*}{c_p}\right) F_2(x, y(x))$$

$$F_2(x, y(x)) = \frac{R_1(-z\sin\varphi_1) F(\theta(z, 0)) \sqrt{-\sin\theta}}{\sqrt{-\sin\theta}\, R_1(-z\sin\varphi_1) F(\theta(z, 0)) F_3(\theta(x, y)) y(x) - 1}$$

$$F_3(\theta(x, y)) = -\frac{3(\sqrt{w}\,\Delta\sin\theta - \sqrt{-\sin\theta})}{2\sin^3\theta\sqrt{-\sin\theta}}, \quad z = \frac{y(x)}{c_p} - x,$$

$$c^* = \frac{dy(x)}{dx}$$

Similarly, from (1.6), (1.7), (2.7) we find the following relations hold in the unloading zone (c_e is the velocity of elastic waves):

$$\Phi_+ = -2\rho_2 a_2^2 c_e w, \quad \Phi_- = 2c\tau_2 \tag{3.5}$$

$$\Phi_\pm = f_1\left(x + \frac{y(x)}{c_e}\right) \pm f_2\left(x - \frac{y(x)}{c_e}\right)$$

$$f_2(x) = \frac{2R_2(-x\sin\varphi_1) g d a_2 - f_1'(x)(g - d)\sin\varphi}{(g + d)\sin\varphi}$$

$$R_2(-x\sin\varphi_1) = -\sin\varphi_1 w'(-x\sin\varphi_1) \quad \text{when} \quad x \leqslant x_p;\ d = \rho_2 a_2\cos\varphi$$
$$g = \rho_1 a_1\cos\varphi_1,\ c_e = -(M^2 - 1)^{-1/2}$$

We assume that on the UW $y = y(x)$ the stresses and displacement rates are continuous, and in this case we have

$$\Phi_+ = -\frac{\rho_2 a_2^2}{8\Delta^2 c_e}\left(\int_0^{\pi+\varphi} \frac{d\theta}{\sqrt{-\sin\theta}} + 2\sqrt{M} - f_3\left(\frac{y(x)}{c_p} - x\right)\right)^2 \tag{3.6}$$
$$\Phi_- = 2ck\cos\theta$$

Differentiating the system of Eqs.(3.6) with respect to x and eliminating the quantities $\partial\theta/\partial x + c^*\partial\theta/\partial y,\ f_3'(y(x)/c_p - x)$ from the equations obtained, we obtain

$$f_1'\left(x + \frac{y(x)}{c_e}\right)\left(1 + \frac{c^*}{c_e}\right) + f_2'\left(x - \frac{y(x)}{c_e}\right)\left(1 - \frac{c^*}{c_e}\right) = \frac{ka_2^2}{c_e c} F_1(\theta(x, y(x)))\left(1 - \frac{c^*}{c_p}\right)\frac{F_2(x, y(x))}{\sqrt{-\sin\theta}} \tag{3.7}$$

$$f_1'\left(x + \frac{y(x)}{c_e}\right)\left(1 + \frac{c^*}{c_e}\right) - f_2'\left(x + \frac{y(x)}{c_e}\right)\left(1 - \frac{c^*}{c_e}\right) = -2ck\sin\theta\left(1 - \frac{c^*}{c_p}\right) F_2(x, y(x)) \tag{3.8}$$

$$f_2'\left(x + \frac{y(x)}{c_e}\right) = \left[2R_2(-\sin\varphi_1 z) g d a_2 - f_1'\left(x + \left(\frac{y(x)}{c_e}\right)\right) \times (g - d)\sin\varphi\right][(g + d)\sin\varphi]^{-1} \tag{3.9}$$

The last equation of (3.9) was obtained from (3.5) for $f_2'(x)$ by changing the argument.

The system of Eqs.(3.7)-(3.9) obtained represents a system of differential equations with divergent argument, for determining the UW $y = y(x)$. It can be solved by numerical methods with the following initial conditions:

$$y(x_p) = 0, \quad y'(x_p) = c_0^*, \quad f_1(x_p) = f_1, \quad f_1'(x_p) = f_1', \quad f_2(x_p) = f_2$$
$$f_2'(x_p) = f_2'$$

Here c_0^* is the initial velocity, UW, and the algorithm for its determination is known /1, 9/. Since the stresses and displacement rates are continuous on UW, therefore the quantities f_1, f_1^1, f_2, f_2' can be found from the solution constructed above in the region of plastic loading.

Moreover, the system of Eqs.(3.7)-(3.9) can be used to determine the velocity of UW at any of its points $(x, y, (x))$. Eliminating the function $f_1'(x + y(x)/c_e)$ from the Eqs.(3.7)-(3.9), we obtain

$$G(c^*) = \left(1 - \frac{c^*}{c_p}\right)\frac{F_2(x, y(x))}{\sqrt{-\sin\theta}}\left(\frac{ka_2^2}{c_e c} F_1(\theta(x, y)) - 2ck\sin\theta\sqrt{-\sin\theta}\right) - 2f_2'\left(x - \frac{y(x)}{c_e}\right)\left(1 - \frac{c^*}{c_e}\right) \tag{3.10}$$

We will assume that the UW is constructed from the data of an arbitrary point at which the velocity of UW is sought. Thus the quantity $y = y(x)$ is known and (3.10) is a linear equation for determining c^*. Therefore, the UW can be constructed by determining c^* from (3.10) at a sequence of points beginning with x_p, and setting up the corresponding segments of the UW by means of the rectilinear segments whose angle of inclination to the x axis is given by c^*. The value of the function $f_2'(x - y(x)/c_e)$, necessary for constructing each successive segment of the UW, is found with help of the relations (3.5), (3.8) from the results of constructing the preceding segments.

Using Eqs.(3.2), (3.5), we can write the function $f_2(x - y(x)/c_e)$ in the form

$$f_2(x + y)(x/c_e) = -\mu_2 (4\Delta^2 c_e)^{-1} F_1^2 (\theta(x, 0)) - ck \cos\theta \tag{3.11}$$

Differentiating (3.11) with respect to x and remembering that $\partial\theta/\partial x < 0$ in the unloading zone, we find that $f_2'(x - (x - x_p)c_p/c_e) \geqslant 0$ in this zone. Therefore, from (3.10) it follows that $G(c_p) \leqslant 0$, $G(c_e) \geqslant 0$, which means that the root of Eq.(3.10) c^* satisfies the condition $|c_p| \leqslant |c_*| \leqslant |c_e|$. Thus we have shown that the velocity of UW at any of its points is not less than the rate of propagation of the plastic waves, and does not exceed the rate of propagation of elastic waves. Therefore the velocity of UW and hence the UW itself, is determined uniquely at every point.

The method of characteristics /9/ can be used instead of the methods given here in constructing the UW.

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